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2005 J. Phys. A: Math. Gen. 38 7773

(http://iopscience.iop.org/0305-4470/38/35/011)

Institute of Physics Publishing Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **38** (2005) 7773–7783 doi:10.1088/0305-4470/38/35/011

# The $sl(2n|2n)^{(1)}$ super-Toda lattices and the heavenly equations as continuum limit

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Received 14 May 2005, in final form 15 May 2005
Published 16 August 2005
Online at stacks.iop.org/JPhysA/38/7773

**Abstract**

The $n \to \infty$ continuum limit of super-Toda models associated with the affine $sl(2n|2n)^{(1)}$ (super)algebra series produces $(2+1)$-dimensional integrable equations in the $\mathbf{S}^1 \times \mathbf{R}^2$ spacetimes. The equations of motion of the (super)Toda hierarchies depend not only on the chosen (super)algebras but also on the specific presentation of their Cartan matrices. Four distinct series of integrable hierarchies in relation with symmetric-versus-antisymmetric, null-versus-nonnull presentations of the corresponding Cartan matrices are investigated. In the continuum limit we derive four classes of integrable equations of heavenly type, generalizing the results previously obtained in the literature. The systems are manifestly $N = 1$ supersymmetric and, for specific choices of the Cartan matrix preserving the complex structure, admit a hidden $N = 2$ supersymmetry. The coset reduction of the (super)-heavenly equation to the $\mathbf{I} \times \mathbf{R}^{(2)} = (\mathbf{S}^1/\mathbf{Z}_2) \times \mathbf{R}^2$ spaceship (with $\mathbf{I}$ a line segment) is illustrated. Finally, integrable $N = 2, 4$ supersymmetrically extended models in $(1+1)$ dimensions are constructed through dimensional reduction of the previous systems.

PACS numbers: 11.30.Pb, 02.30.Ik

## 1. Introduction

The class of the so-called heavenly equations has been introduced by Plebański [1] to describe the solutions of the self-dual Einstein gravity. In this context the heavenly equations have been widely analysed, their relation with the $SU(\infty)$ Toda field equations being pointed out (see, e.g., [2] and the references therein).

The integrability properties of such equations were investigated in a series of papers (see [3]), while, at the beginning of the nineties, it was further proven [4] that the first heavenly equation describes the string theory with local $N = 2$ supersymmetry (for a recent review containing an updated list of references see [5]). The connection with superstrings provides a strong physical motivation for the construction of the integrable supersymmetric extensions of the heavenly equations (to be recovered from superalgebraic data).

For what concerns such supersymmetric extensions, the present state of the art is as follows. Saveliev and Sorba [6] first derived the $(2+1)$-dimensional $N = 1$ supersymmetric heavenly equation as a continuum limit (for $n \to \infty$) of a discretized super-Toda system based on the finite $sl(n|n+1)$ superalgebra series. Discretized super-Toda systems were further investigated in several papers [7]. The systems analysed in [8] admit a hidden $N = 2$ supersymmetry. In [9] it was shown, following [10], how to exploit the complex structure of $sl(n|n+1)$ to encode data in a way to construct an integrable, manifestly $N = 2$ supersymmetric, Toda lattice admitting an $N = 2$ superheavenly equation in its continuum limit (the superheavenly equation of [6] being recovered as a special reduction).

In this paper we address two separate issues and present their solutions. In this way we are able to enlarge the class of integrable supersymmetric Toda lattices (as well as their continuum limit) produced in the literature with the introduction of new sets of integrable equations.

We recall first the interpretation of the dots of the (super)algebra Dynkin diagrams as discretized positions of the (super)Toda lattice which label, in the continuum limit, an extra dimension. For the class of (super)Toda lattices based on *finite* (super)Lie algebras, as the ones studied in [6, 9], the discretized extra dimension corresponds to the half-line $\mathbf{R}_+ = [0, +\infty]$. The continuum limit of the superToda lattice therefore produces a $(2+1)$-dimensional integrable system based on either $\mathbf{R}_+ \times \mathbf{R}^{(2,0)}$ or $\mathbf{R}_+ \times \mathbf{R}^{(1,1)}$ (according to whether the (super)Toda lattice is formulated as a relativistic two-dimensional Euclidean or Minkowskian theory). On the other hand, due to cyclicity of the affine (super)Lie algebra Dynkin diagrams introduced below, their dots produce a discretization of the $\mathbf{S}^1$ circle. A discretized (super)Toda lattice based on, e.g., the $sl(2n|2n)^{(1)}$ affine superalgebra series, see [11, 12], produces in the continuum limit a class of $(2+1)$ integrable systems whose base spacetimes are either $\mathbf{S}^1 \times \mathbf{R}^{(2,0)}$ or $\mathbf{S}^1 \times \mathbf{R}^{(1,1)}$. It can be further proven that a coset construction $\mathbf{I} = \mathbf{S}^1/\mathbf{Z}_2$ allows us to express the continuum limit in the $\mathbf{I} \times \mathbf{R}^{(2,0)}$ or $\mathbf{I} \times \mathbf{R}^{(1,1)}$ spacetimes, where $\mathbf{I}$ is a closed interval which, without loss of generality, can be chosen to be given by $\mathbf{I} = [0, 1]$.

In the present paper, unlike the previous works in the literature based on *finite* superalgebras, we construct superToda lattices associated with the $sl(2n|2n)^{(1)}$ *affine* superalgebras (the reason for us to work with this special class of $n$-parametric affine superalgebras is the presence, see [12], of a complex structure; due to that, and depending on the given chosen construction used in the following, in special cases we are able to obtain $N = 2$ supersymmetric systems), producing in the $n \to \infty$ continuum limit the $\mathbf{S}^1 \times \mathbf{R}^2$ systems previously discussed.

It should be remembered that a (super)Lax pair of a two-dimensional (super)Toda system only requires the algebraic data contained in the given, associated, Cartan matrix. The second main issue addressed in this paper concerns the fact that, however, the (super)Toda model is *not* uniquely specified by its associated Cartan matrix. An algebraically equivalent, but a different presentation for the Cartan matrix, actually produces a different type of Toda model. This feature often passes unnoticed and its importance not recognized, implying that whole classes of integrable systems are not correctly identified and are instead disregarded.

For super-algebras it is already well known that we need to work with the specific presentation of Dynkin diagrams realized by *all* fermionic simple roots (whenever this is indeed possible, following the superalgebras classification, see [11, 12]), in order to have a linearly implemented, manifest supersymmetric Toda system of equations. The other Dynkin superalgebra presentations, according to [13], produce spontaneously broken, nonlinearly realized, supersymmetric Toda models.

However, a given presentation of the Cartan matrix is already important for the derivation of Toda models associated with bosonic Lie algebras (as well as for super-Toda models derived from Cartan matrices associated with *all* fermionic simple roots Dynkin diagrams). It is known [14] that, without loss of generality, the Cartan matrix can be chosen, e.g., either symmetric or antisymmetric. The derivation of the bosonic heavenly equation as the $n \to \infty$ continuum limit of the $sl(n)$ Toda lattice *requires* the symmetric presentation of the $sl(n)$ Cartan matrix (given by $A_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$) which makes it correspond to a discretization of the second-order derivative entering the bosonic heavenly equation [3].

For what concerns the $sl(2n|2n)^{(1)}$ affine superalgebra series, see e.g. [12], we can work within a purely fermionic simple roots cyclic Dynkin diagram. Four main classes of presentations for the associated Cartan matrices $A_{ij}$ can be considered. They can be obtained by combining

(i) the symmetric presentation for the Cartan matrix or,
(ii) its antisymmetric counterpart, as well as,

(a) the 'null' presentation (i.e. for any $i$, $\sum_j A_{ij} = 0$ and, similarly, for any $j$, $\sum_i A_{ij} = 0$) or,
(b) the 'nonnull' presentation (such that $\sum_j A_{ij} \neq 0$, $\sum_i A_{ij} \neq 0$, $|\sum_j A_{ij}| = |\sum_i A_{ij}| = 2$).

For each given $n$, as well as in the $n \to \infty$ continuum limit, four corresponding classes of supersymmetric systems of integrable equations, labelled (ia), (ib), (iia) and (iib) respectively, are produced accordingly.

For us it is convenient to work with the $N = 1$ superfield formalism of [15], rather than the $N = 2$ formalism introduced in [10] and also used in [9]. The reason is that the $N = 1$ formalism is more general. Due to the existence of a complex structure, if the latter is preserved by the specific presentation of the Cartan matrix, the associated Toda system automatically admits a hidden $N = 2$ supersymmetry. On the other hand, the $N = 2$ formalism does not allow us to reproduce the super-Toda systems which possess only an $N = 1$ supersymmetry. In the following we construct a whole new class of $N = 1$ super-Toda models.

The Toda models analysed in this paper (for any value of $n = 1, 2\ldots$ and in the limit $n \to \infty$, as well as for each one of the four presentations of the Cartan matrices) are all given by a coupled nonlinear system of two $N = 1$ superfields, plus two extra superfields satisfying equations in the background produced by the first two coupled superfields.

The scheme of the paper is as follows. In the next section we introduce, following [15], the $N = 1$ Lax pair formulation for super-Todas and apply it to derive the four models associated with the $sl(2|2)^{(1)}$ superalgebra. In section 3 we extend these results to any $n$, producing the $sl(2n|2n)^{(1)}$ super-Toda lattices and further deriving their continuum limit (corresponding to the four classes of super-heavenly equations). In section 4 we show how to reduce a (super)-heavenly type of equation from $\mathbf{S}^1 \times \mathbf{R}^2$ to its coset $\mathbf{I} \times \mathbf{R}^2$, with $\mathbf{I} = \mathbf{S}^1/\mathbf{Z}_2$. In section 5 we discuss the $(1+1)$-dimensional reductions of the previous systems, producing super-hydrodynamical types of equations with $N = 1, 2, 4$ supersymmetries. In the conclusion we address some open problems concerning the construction of $N > 2$, $N$-extended supersymmetric two-dimensional Toda models and $(2 + 1)$-dimensional heavenly types of equation.

## 2. The $sl(2|2)^{(1)}$ super-Toda models

Let us introduce first, following [15], the manifest $N = 1$ formalism which will be employed throughout this paper. The fermionic derivatives are given by

$$D_\pm = \frac{\partial}{\partial\theta_\pm} + \theta_\pm\partial_\pm, \tag{2.1}$$

where $\theta_\pm$ are fermionic Grassmann coordinates, while $x_\pm$ are the light-cone coordinates $\left(\partial_\pm = \frac{\partial}{\partial x_\pm}\right)$ s.t. $x_\pm = x \pm t$ in the Minkowskian case and $x_\pm = x \pm \mathrm{i}t$ in the Euclidean (in the latter case $x_+ = x_-{}^*$).

The $N = 1$ supersymmetric Lax pairs are given by

$$L_+ = D_+\Phi + \mathrm{e}^{\Phi}F_+\mathrm{e}^{-\Phi}, \qquad L_- = -D_-\Phi + \mathrm{e}^{-\Phi}F_-\mathrm{e}^{\Phi}, \tag{2.2}$$

where

$$\Phi = \frac{1}{2}\sum_j \Phi_j H_j, \qquad F_+ = \sum_j F_{+j}, \qquad F_- = \sum_j F_{-j}. \tag{2.3}$$

In the above formula '$\Phi_j$' denotes a set of ($N = 1$) bosonic superfields; the sums over $j$ are restricted to, respectively, the Cartan generators and the simple (positive and negative) fermionic roots. For our purposes we only need to know the following algebraic relations between Cartan generators and fermionic simple roots, given by

$$[H_i, F_{\pm j}] = \pm A_{ij}F_{\pm j}, \qquad \{F_i, F_{-j}\} = \delta_{ij}H_j, \tag{2.4}$$

where $A_{ij}$ denotes the Cartan matrix.

The zero-curvature equation, given by,

$$\{D_+ + L_+, D_- + L_-\} = 0 \tag{2.5}$$

reproduces the set of super-Toda equations (the sum over repeated indices is understood)

$$D_+D_-\Phi_j = \exp(\Phi_i A_{ij}). \tag{2.6}$$

Let us specialize now the above system to the four specific presentations for the $sl(2|2)^{(1)}$ Cartan matrix discussed in the introduction. They are respectively given by [11], [12]

$$(\mathrm{ia}) \equiv \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \tag{2.7}$$

$$(\mathrm{ib}) \equiv \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \tag{2.8}$$

$$(\mathrm{iia}) \equiv \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \tag{2.9}$$

$$(\mathrm{iib}) \equiv \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}. \tag{2.10}$$

It is perhaps worth showing the explicit generator redefinitions which allow us to connect the different presentations of the Cartan matrices. We have

(1) The (ia) $\rightarrow$ (ib) transformation corresponds to the mappings $H_3 \rightarrow -H_3$, $H_4 \rightarrow -H_4$, $F_{\pm 3} \rightarrow \mathrm{i}F_{\mp 3}$, $F_{\pm 4} \rightarrow \mathrm{i}F_{\mp 4}$ (the other generators are left unchanged),
(2) the (ib) $\rightarrow$ (iia) transformation is obtained through $H_2 \rightarrow -H_2$, $H_3 \rightarrow -H_3$, $F_{\pm 3} \rightarrow \mathrm{i}F_{\mp 3}$, $F_{\pm 2} \rightarrow \mathrm{i}F_{\pm 2}$, $F_{\pm 4} \rightarrow F_{\mp 4}$ (the other generators are unchanged),
(3) the (ib) $\rightarrow$ (iib) transformation corresponds to $H_2 \rightarrow -H_2$, $H_4 \rightarrow -H_4$, $F_{\pm 2} \rightarrow \mathrm{i}F_{\pm 2}$, $F_{\pm 4} \rightarrow \mathrm{i}F_{\pm 4}$ (the other generators are unchanged).

Before writing the four sets of super-Toda equations corresponding to the $sl(2|2)^{(1)}$ affine superalgebra, let us set first

$$\Phi_\pm = \tfrac{1}{2}(\Phi_3 \pm \Phi_1), \qquad \Lambda_\pm = \tfrac{1}{2}(\Phi_4 \pm \Phi_2). \tag{2.11}$$

We have

*Case (ia)*. This presentation of the $sl(2|2)^{(1)}$ Cartan matrix was employed in [10]. The Cartan matrix is degenerate (with rank two) and the system is not conformally invariant. In [10], in order to remove the degeneration of the Cartan matrix, the $sl(2|2)^{(1)}$ superalgebra was extended with the addition of two extra Cartan generators. The system discussed in [10] was based on the conformally affine approach pioneered in [16] and proved to be a conformal extension of the $N = 2$ sinh-Gordon equation with a spontaneous breaking of the conformal symmetry. For the non-extended $sl(2|2)^{(1)}$ affine superalgebra we get here

$$D_+D_-\Phi_- = \sinh(2\Lambda_-), \qquad D_+D_-\Lambda_- = \sinh(2\Phi_-), \tag{2.12}$$

together with

$$D_+D_-\Phi_+ = \cosh(2\Lambda_-), \qquad D_+D_-\Lambda_+ = \cosh(2\Phi_-). \tag{2.13}$$

This system of equations corresponds to two coupled sinh-Gordon equations plus two background equations. The complexification of the coupled sinh-Gordon superfields induces a hidden $N = 2$ supersymmetry. Indeed, the second supersymmetry closing the $N = 2$ invariance is explicitly given by the transformations (associated with the light-cone coordinates $x+$, $x_-$, respectively)

$$\begin{aligned} \delta_+\Phi_\pm &= \mathrm{i}\epsilon D_+\Phi_\pm, & \delta_+\Lambda_\pm &= -\mathrm{i}\epsilon D_+\Lambda_\pm, \\ \delta_-\Phi_\pm &= \mathrm{i}\epsilon D_-\Phi_\pm, & \delta_-\Lambda_\pm &= -\mathrm{i}\epsilon D_-\Lambda_\pm. \end{aligned} \tag{2.14}$$

A consistent $N = 1$ reduction corresponds to identifying $\Phi_\pm = \Lambda_\pm$ (in the minimal case $\Phi_\pm$ can be assumed to be a *real* superfield), while a consistent $N = 2$ reduction corresponds to setting $\Phi_\pm = \Lambda_\pm{}^*$ for a *complex* superfield $\Phi_\pm$.

*Case (ib)*. We get

$$D_+D_-\Phi_+ = \exp(2\Lambda_+), \qquad D_+D_-\Lambda_+ = \exp(2\Phi_+), \tag{2.15}$$

together with

$$D_+D_-\Phi_- = 0, \qquad D_+D_-\Lambda_- = 0. \tag{2.16}$$

This system of equations corresponds to two coupled (super)Liouville equations plus two free equations. As before, the complexification of the coupled Liouville superfields gives an $N = 2$ supersymmetry. We have a consistent $N = 1$ reduction, given by $\Phi_\pm = \Lambda_\pm$, and a consistent $N = 2$ reduction for $\Phi_\pm = \Lambda_\pm{}^*$.

*Case (iia)*

$$D_+D_-\Phi_- = \sinh(-2\Lambda_-), \qquad D_+D_-\Lambda_- = \sinh(2\Phi_-), \tag{2.17}$$

together with

$$D_+D_-\Phi_+ = \cosh(2\Lambda_-), \qquad D_+D_-\Lambda_+ = \cosh(2\Phi_-). \tag{2.18}$$

This system of equations corresponds to two coupled sinh-Gordon equations with 'wrong' sign plus two background equations. The 'wrong' sign means that $\Phi_\pm = \Lambda_\pm$ is *not* a consistent reduction and that the complexification of the above coupled sinh-Gordon superfields *does not* produce an $N = 2$ supersymmetry.

*Case (iib)*

$$D_+D_-\Phi_+ = \exp(-2\Lambda_+), \qquad D_+D_-\Lambda_+ = \exp(2\Phi_+), \tag{2.19}$$

together with

$$D_+D_-\Phi_- = 0, \qquad D_+D_-\Lambda_- = 0. \tag{2.20}$$

This system of equations corresponds to two coupled Liouville equations with 'wrong' sign plus two free equations. As before, the 'wrong' sign means that $\Phi_\pm = \Lambda_\pm$ is *not* a consistent reduction and that the complexification of the above coupled Liouville superfields *does not* produce an $N = 2$ supersymmetry.

## 3. The affine $sl(2n|2n)^{(1)}$ super-Toda lattices and their continuum limit

Let us now discuss the different classes of $sl(2n|2n)^{(1)}$ super-Toda lattices and their continuum limits. We work with the fermionic presentation of the simple roots, the Dynkin diagram being cyclic and admitting $4n$ dots. The four associated presentations of the Cartan matrices are given, for $k, l = 1, \ldots, 4n$, by

$$\begin{aligned}
&\text{(ia)} \quad A_{kl} = -\delta_{1,k}\delta_{l,4n} - \delta_{k,4n}\delta_{l,1} + (-1)^k\delta_{k,l+1} - (-1)^k\delta_{l,k+1},\\
&\text{(ib)} \quad A_{kl} = \delta_{1,k}\delta_{l,4n} + \delta_{k,4n}\delta_{l,1} + \delta_{k,l+1} + \delta_{l,k+1},\\
&\text{(iia)} \quad A_{kl} = -\delta_{1,k}\delta_{l,4n} + \delta_{k,4n}\delta_{l,1} - \delta_{k,l+1} + \delta_{l,k+1},\\
&\text{(iib)} \quad A_{kl} = \delta_{1,k}\delta_{l,4n} - \delta_{k,4n}\delta_{l,1} - (-1)^k\delta_{k,l+1} - (-1)^k\delta_{l,k+1}.
\end{aligned} \tag{3.1}$$

For our purposes it is convenient to group the superfields as follows, where $i$ plays the role of a discretized extra-time variable,

$$\begin{aligned}
B_i &= \Phi_{2i} - \Phi_{2i-2}, \qquad & C_i &= \Phi_{2i} + \Phi_{2i-2},\\
E_i &= \Phi_{2i+1} - \Phi_{2i-1}, \qquad & F_i &= \Phi_{2i+1} + \Phi_{2i-1}.
\end{aligned} \tag{3.2}$$

The cyclicity condition is assumed; namely, for any $V_i \equiv B_i, C_i, E_i, F_i$ we get $V_{i+2n} = V_i$.

In the limit $n \to \infty$ the discretized coordinate $i$ describes the compactified coordinate $\tau \in [0, 2\pi R]$, through

$$\begin{aligned}
&V_i \sim V(\tau)\\
&\frac{V_{i+1} - V_i}{\Delta} \sim \frac{\partial}{\partial\tau}V, \qquad \text{with} \quad \Delta = \frac{2\pi R}{2n}.
\end{aligned}$$

By implementing the $N = 1$ Lax pair system of [15] discussed in the previous section, we get the following systems of super-Toda equations:

(ia) case, given by

$$\begin{aligned}
D_+D_-B_i &= \exp(E_i) - \exp(E_{i-1}),\\
D_+D_-C_i &= \exp(E_i) + \exp(E_{i+1}),\\
D_+D_-E_i &= \exp(-B_{i+1}) - \exp(-B_i),\\
D_+D_-F_i &= \exp(-B_{i+1}) + \exp(-B_i).
\end{aligned} \tag{3.3}$$

By conveniently reabsorbing the $\Delta$ factor in the redefinition of $\tau$, we get in the continuum limit

$$D_+D_-B = \frac{\partial}{\partial\tau}\exp(E), \qquad D_+D_-E = \frac{\partial}{\partial\tau}\exp(-B) \tag{3.4}$$

together with

$$D_+D_-C = 2\exp(E), \qquad D_+D_-F = 2\exp(-B). \tag{3.5}$$

It corresponds to two coupled heavenly equations with the 'wrong' sign, plus two infinite sets (labelled by $\tau$) of equations of Liouville type in the heavenly background.

(ib) case, given by

$$\begin{aligned} D_+D_-B_i &= \exp(F_i) - \exp(F_{i-1}), \\ D_+D_-C_i &= \exp(F_i) + \exp(F_{i-1}), \\ D_+D_-E_i &= \exp(C_{i+1}) - \exp(C_i), \\ D_+D_-F_i &= \exp(C_{i+1}) + \exp(C_i). \end{aligned} \tag{3.6}$$

The continuum limit gives us

$$D_+D_-C = 2\exp(F), \qquad D_+D_-F = 2\exp(C) \tag{3.7}$$

together with

$$D_+D_-B = \frac{\partial}{\partial\tau}\exp(F), \qquad D_+D_-E = \frac{\partial}{\partial\tau}\exp(C).$$

It corresponds to two infinite ($\tau$-dependent) series of coupled Liouville equations with the 'good' sign. $D = F$ is a consistent $N = 1$ reduction, while the complexified system admits an $N = 2$ supersymmetry and $D = F^*$ is a consistent $N = 2$ reduction. The remaining two equations are of heavenly type in the Liouville background.

(iia) case, given by

$$\begin{aligned} D_+D_-B_i &= \exp(-E_i) - \exp(-E_{i-1}), \\ D_+D_-C_i &= \exp(-E_i) + \exp(-E_{i-1}), \\ D_+D_-E_i &= \exp(-B_{i+1}) - \exp(-B_i), \\ D_+D_-F_i &= \exp(-B_{i+1}) + \exp(-B_i). \end{aligned} \tag{3.9}$$

The continuum limit gives us

$$D_+D_-B = \frac{\partial}{\partial\tau}\exp(-E), \qquad D_+D_-E = \frac{\partial}{\partial\tau}\exp(-B)$$

together with

$$D_+D_-C = 2\exp(-E), \qquad D_+D_-F = 2\exp(-B).$$

It corresponds to two coupled Heavenly equations with the 'good' sign. $B = E$ is a consistent $N = 1$ reduction, while the complexified system admits $N = 2$ supersymmetry and $B = E^*$ is a consistent $N = 2$ reduction. This is the $N = 2$ heavenly equation system introduced in [9]. The two remaining equations are of Liouville type in the heavenly background.

(iib) case, given by

$$\begin{aligned} D_+D_-B_i &= \exp(F_i) - \exp(F_{i-1}), \\ D_+D_-C_i &= \exp(F_i) + \exp(F_{i-1}), \\ D_+D_-E_i &= \exp(-C_{i+1}) - \exp(-C_i), \\ D_+D_-F_i &= \exp(-C_{i+1}) + \exp(-C_i). \end{aligned} \tag{3.12}$$

The continuum limit gives us

$$D_+D_-C = 2\exp(F), \qquad D_+D_-F = 2\exp(-C) \tag{3.13}$$

together with

$$D_+D_-B = \frac{\partial}{\partial\tau}\exp(F), \qquad D_+D_-E = \frac{\partial}{\partial\tau}\exp(-C). \tag{3.14}$$

It corresponds to two infinite ($\tau$-dependent) sets of coupled Liouville equations with the 'wrong' sign (the system is $N = 1$ supersymmetric only and there is no consistent reduction), plus two equations of heavenly type in the Liouville background.

## 4. The $\mathbf{S}^1/\mathbf{Z}_2$ coset construction

Let us analyse specifically the closed reduced system for the (iia) case. We have the system of super-Toda equations

$$\begin{aligned} D_+D_-B_i &= \exp(-E_i) - \exp(-E_{i-1}), \\ D_+D_-E_i &= \exp(-B_{i+1}) - \exp(-B_i). \end{aligned} \tag{4.1}$$

It is convenient to set

$$B_i = A_{2i}, \qquad E_i = A_{2i+1} \tag{4.2}$$

in order to re-express the above system as

$$\begin{aligned} D_+D_-A_{2i} &= \exp(-A_{2i+1}) - \exp(-A_{2i-1}), \\ D_+D_-A_{2i+1} &= \exp(-A_{2i+2}) - \exp(-A_{2i}). \end{aligned} \tag{4.3}$$

We can write, compactly,

$$D_+D_-A_j = \exp(-A_{j+1}) - \exp(-A_{j-1}), \tag{4.4}$$

where $j$ ($A_{j+4n} = A_j$, see the previous section discussion) represents the discretization of the angular coordinate $\theta_j = \frac{2\pi j}{4n}$. In the continuum limit this system of equations reads as follows:

$$D_+D_-A = \frac{\partial}{\partial\tau}\exp(-A). \tag{4.5}$$

The (4.4) discrete system, besides the $\mathbf{S}^1$ continuum limit ('$j$' corresponds to the point labelled by $x_j = R\cos\theta_j$, $y_j = R\sin\theta_j$) provides a discretization of the $\mathbf{I} = [-R, R]$ line since one can consistently project the associated $A_j$ (super)fields onto the circle diameter, by setting $A(x_j, y_j) \equiv A(x_j, -y_j)$ for $(x_j, y_j) \mapsto x_j$. Indeed, it is easily proven that the constraint

$$A_{4n-j}(x_+, \theta_+, x_-, \theta_-) = A_j(x_+, \theta_+, -x_-, -\theta_-) \tag{4.6}$$

is consistent w.r.t. the (4.4) equations of motion. Please note the change in sign in front of $x_-, \theta_-$, introduced in order to compensate for the presence of a minus sign on the rhs of (4.4).

The above constrained system corresponds to a discretized version of the supersymmetric heavenly equation in the interval $\mathbf{I}$, with the superfields labelled by $A_0$ and $A_{2n}$ corresponding to the initial and final positions.

## 5. The $N = 2, 4$ super-hydrodynamical reductions in (1 + 1) dimensions

The nonlinear systems in (1 + 2) dimensions derived in the previous section can be dimensionally reduced by setting $x_+ = x_- = x$. These reduced (1 + 1)-dimensional systems inherit the integrability properties of their higher-dimensional analogues. In [9], it was shown that these types of nonlinear equations can be recast as super-hydrodynamical types of equations, generalizing the bosonic hydrodynamical results of [17].

The (1 + 1)-dimensionally reduced equations are supersymmetric, with twice as many supersymmetries as their corresponding (1+2)-dimensional systems ($N = 1 \to N = 2$, $N = 2 \to N = 4$). The $N = 4$ systems are obtained in correspondence with the (complexified) (ib) and (iia) cases. The $N = 4$ generators, satisfying

$$\{Q_i, Q_j\} = -2\delta_{ij}\partial_x \tag{5.1}$$

are explicitly given by

$$\begin{aligned} Q_1 &= \tfrac{\partial}{\partial\theta_+} - \theta_+\partial_x, & Q_2 &= \tfrac{\partial}{\partial\theta_-} - \theta_-\partial_x, \\ Q_3 &= \mathrm{i}\left(\frac{\partial}{\partial\theta_+} + \theta_+\partial_x\right), & Q_4 &= \mathrm{i}\left(\frac{\partial}{\partial\theta_-} + \theta_-\partial_x\right). \end{aligned} \tag{5.2}$$

It is convenient to explicitly present one of these systems (the (iia)) in components.

Let us have

$$\begin{aligned} B &= b + \theta_+\psi_+ + \theta_-\psi_- + \theta_+\theta_- a, \\ C &= c + \theta_+\xi_+ + \theta_-\xi_- + \theta_+\theta_- d, \end{aligned} \tag{5.3}$$

with complex, $a, b, c, d$ bosonic and $\psi_\pm$, $\xi_\pm$ fermionic fields, all dependent on $x$ and $\tau$. We further set the consistent constraints $E = B^*$, $F = C^*$. The (1 + 1)-dimensionally reduced, type (iia) system is equivalent to the equations

$$\begin{aligned} a &= -\frac{\partial}{\partial\tau}\,\mathrm{e}^{-b^*}, \qquad \partial_x\psi_\pm = \pm\frac{\partial}{\partial\tau}(\mathrm{e}^{-b^*}\psi_\mp{}^*), \\ \partial_x{}^2 b &= -\frac{\partial}{\partial\tau}[\mathrm{e}^{-b^*}(a^* + \psi_+{}^*\psi_-{}^*)], \end{aligned} \tag{5.4}$$

together with the background equations

$$\begin{aligned} d &= -2\mathrm{e}^{-b^*}, \qquad \partial_x\xi_\pm = \pm 2(\mathrm{e}^{-b^*}\psi_\mp{}^*), \\ \partial_x{}^2 c &= 2\mathrm{e}^{-b^*}\left(\frac{\partial}{\partial\tau}\,\mathrm{e}^{-b} - \psi_+{}^*\psi_-{}^*\right) \end{aligned} \tag{5.5}$$

($a$ and $d$ are the auxiliary fields).

## 6. Conclusions

In this work we constructed a set of $N = 1$ and $N = 2$ supersymmetric (2 + 1)-dimensional integrable systems arising as a continuum limit of super-Toda models obtained from the affine $sl(2n|2n)^{(1)}$ superalgebras series, in connection with different presentations for the associated Cartan matrices. The equations, of heavenly type, were derived by using a manifest $N = 1$ superLax formalism and involved in all cases under considerations two coupled superfields and two extra superfields in the background produced by the two previous ones. These systems can easily be extended, *still preserving integrability*, by making the associated superfields valued not only on the real and complex numbers, but, e.g., on quaternions as well. As an example, the $N = 2$ (iia) coupled system of equations, extended to the quaternionic superfields

$B = B_0 + \sum_{i=1,2,3} e_i B_i$, $E = E_0 + \sum_{i=1,2,3} e_i E_i$ (with $e_i$'s the three imaginary quaternions satisfying $e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k$) reads as follows, in terms of the real superfields

$$\begin{aligned} D_+D_-B_0 &= \frac{\partial}{\partial\tau}(\exp(-E_0)\cos(\mathcal{E})), \\ D_+D_-B_i &= -\frac{\partial}{\partial\tau}\left(\exp(-E_0)\frac{\sin(\mathcal{E})}{\mathcal{E}}E_i\right), \\ D_+D_-E_0 &= \frac{\partial}{\partial\tau}(\exp(-B_0)\cos(\mathcal{B})), \\ D_+D_-E_i &= -\frac{\partial}{\partial\tau}\left(\exp(-B_0)\frac{\sin\mathcal{B}}{\mathcal{B}}B_i\right), \end{aligned} \tag{6.1}$$

where $\mathcal{E} = \sqrt{E_1^2 + E_2^2 + E_3^2}$ and $\mathcal{B} = \sqrt{B_1^2 + B_2^2 + B_3^2}$.

These equations inherit the integrability property from the corresponding (super)Lax presentation which can automatically accommodate quaternionic-valued superfields. It is worth noting that, despite the fact that the bosonic sector of $sl(2n|2n)^{(1)}$ is quaternionic (see [18]), the above system of equations is only $N = 2$ supersymmetric; therefore, the *quaternionization* does not induce any hidden $N = 4$ supersymmetry, while the *complexification*, in those selected cases that have been discussed here, can induce a hidden $N = 2$.

An extended $N = 4$ supersymmetry is only recovered when performing the $(1 + 2) \mapsto (1 + 1)$ dimensional reduction, see (5.4).

Perhaps one of the most challenging open problems concerns the existence of integrable $N > 2$ supersymmetric extensions of the heavenly equation directly in $(1+2)$ dimensions. It is unclear whether, let us say an $N = 4$, integrable heavenly equation indeed exists. This problem seems to be linked with the possibility of reformulating the $N = 4$ version of the Liouville equation (constructed [19] in the eighties in terms of quaternionic harmonic superfields) as a super-Toda system derived from a super-Lax based on a given, correctly identified, Cartan matrix superalgebra. To our knowledge, this is still an unsolved problem which deserves being addressed.

## Acknowledgments

One of us (FT) is grateful for the hospitality at the Institute of Theoretical Physics of the University of Wrocław, where this work was initiated.

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